Topology and Physics 2018 - lecture 6

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6.1 Associated bundles and principal bundles

6.1.1 Why vector bundles?

In the previous lecture, vector bundles were introduced: spaces that look like a (base) manifold which "at every point has a vector space attached" – the canonical example being the tangent bundle, consisting of all tangent vectors at all points of a given manifold. Each vector bundle has three types of spaces attached to it: a *total space* E, a *base space* M, and over every point $x \in M$ the *fiber* over that point, F_x. In our canonical example, M would be the space-time manifold we are interested in, F_x would be the vector space of all tangent vectors of M at x, and E would be the collection of all F_x together.

Of course, there is some asymmetry in this construction: M and E are rather general manifolds, but the F_x are restricted to be *vector spaces*. This restriction is not needed at all: one can straightforwardly generalize the construction of a vector bundle to a *fiber bundle*, where the fiber F_x can be a more general manifold. Of course, with more generality comes the loss of structure, and so completely general fiber bundles are not of that much interest to the physicist. However, we may be interested in replacing the "vector space structure" of the fiber by some other interesting structure, such as that of a Lie group – and it turns out that in particular this structure leads to some very interesting bundles in physics. In this section, we will introduce these bundles – so called *principal bundles* – but to do so, we first need to know a bit more about the concept of *associated bundles*.

6.1.2 Associated bundles

Recall from the previous lecture that there are two ways to think about vector bundles: we have what we might call the "top-down approach" and the "bottom-up approach". In the top-down approach, the total space E is given, and we have a projection $\pi : E \to M$ to our base manifold, such that for every $x \in M$ the fiber over x, $F_x \equiv \pi^{-1}(x)$ is isomorphic to some "canonical fiber" vector space F. (Of course, there are then some restrictions like local triviality that make this structure into an actual vector bundle; these restrictions were discussed in the previous lecture.)

In the "bottom-up approach", E is actually *constructed*, by starting from a manifold M, a cover $\{U_i\}$ of that manifold, and a canonical fiber F. We then first construct trivial bundles over the sets U_i: $E_i = U_i \times F$, and then we glue E_i to E_j whenever U_i and U_j overlap: $(x, v_i) \in E_i$ is identified with $(x, \phi_{ji}(x)v_i) \in E_j$. (Again, details can be found in the notes for the previous lecture.) To make the different constructions concrete, let us consider two very simple bundles: the two different real line bundles over the circle $M = S^1$. One is the cylinder (the trivial bundle $S^1 \times \mathbb{R}$); the other is the Möbius strip.

- In the top-down construction, the entire E is constructed first. This can be done for example by starting from $\mathbb{R} \times \mathbb{R}$, and noticing that S^1 can be obtained from the first $\mathbb{R}$ factor by identifying $x \in \mathbb{R}$ with $x + 1$. Thus, the simplest construction is to look at $E = \mathbb{R} \times \mathbb{R}/\sim$ where $(x, y) \sim (x + 1, y)$. The resulting space is clearly the cylinder. However, one might also take a different equivalence relation where $(x, y) \sim (x + 1, -y)$; then $E = \mathbb{R} \times \mathbb{R}/\sim$ is the Möbius strip. (It is a useful exercise to prove that these are essentially the only two constructions involving a linear transformation acting on the second $\mathbb{R}$ factor: that for example $(x, y) \sim (x + 1, \alpha y)$ leads to a space which is topologically equivalent to the cylinder if $\alpha > 0$ and to the Möbius strip if $\alpha < 0$.)

- In the bottom-up approach, we first take a cover of the circle, thought of as the interval $[0, 1]$ with the end points identified. For example, we could take $A = (-\epsilon, \frac{1}{2} + \epsilon)$ and $B = (\frac{1}{2} - \epsilon, 1 + \epsilon)$ with some small ϵ, and then construct the trivial bundles $E_A = A \times \mathbb{R}$ and $E_B = B \times \mathbb{R}$. Then we glue together the two overlaps: we identify $(x, y) \in E_A$ with $(x, y) \in E_B$ for x near $\frac{1}{2}$, and we identify $(x, y) \in E_A$ with $(x + 1, \pm y) \in E_B$ for x near 0. Again, the sign determines which space we get: the plus sign leads to the cylinder, whereas the minus sign leads to the Möbius strip. Again, it is a useful exercise to show that these are essentially the only possibilities, and that glueing y to αy with $\alpha \in \mathbb{R}^*$ leads to one of these two topologies.

Even though the top-down construction may seem a bit more elegant mathematically (as no glueing is required), it is actually the bottom-up construction which makes it rather easy to replace bundles by closely related bundles. The reason for this is as follows: suppose that M and its cover $\{U_i\}$ are fixed once and for all, and that $x \in U_i \cap U_j$. Then we are instructed to glue

$$U_i \times F \ni (x, v_i) \sim (x, \phi_{ji}(x)v_i) \in U_j \times F \tag{6.1}$$

where

$$\phi_{ji}(x) \in GL(n, \mathbb{R}) \qquad \text{or} \qquad \phi_{ji}(x) \in GL(n, \mathbb{C}) \tag{6.2}$$

depending on whether F is a real or complex vector space. Here, of course, n is the dimension of F. However, very often, we do not need the full $GL(n, \mathbb{R})$ to glue bundles together; for example, in the example above we did not quite need $GL(1, \mathbb{R}) = \mathbb{R}^*$ to glue the fibers; $\mathbb{Z}_2 = \{-1, 1\}$ was sufficient. Thus, we can often glue using a much smaller group

$$G \subset GL(n, \mathbb{R}) \qquad \text{or} \qquad G \subset GL(n, \mathbb{C}). \tag{6.3}$$

G may be a discrete group, as above, but we are typically interested in the slightly more generic situation where G itself is a matrix Lie group, such als $SO(n)$, $U(n)$, $SU(n)$, and so on. The group G is called the *structure group* of the bundle.

Now of course, a Lie group like $SO(n)$ can act on n-dimensional vectors, but there are many other representations of $SO(n)$ (or any other structure group G) as well. Let's say we have some other m-dimensional represtation R, that is: for $g \in G$, $R[g]$ is some $m \times m$ matrix acting on an m-dimensional vector space V, and the map $g \to R[g]$ is such that

$$R[gh] = R[g]R[h]. \tag{6.4}$$

Then it is not very hard to check that we can construct a new vector bundle as follows: start from trivial bundles $\hat{E}_i = U_i \times V$ and glue these together as

$$U_i \times V \ni (x, w_i) \sim (x, R[\phi_{ji}(x)]w_i) \in U_j \times V. \tag{6.5}$$

It is not hard to check that this construction satisfies all the rules for the bottom-up construction of a vector bundle, and therefore this gives a new bundle $\hat{E}$ which still has base space M, but now has fiber V. E and $\hat{E}$ are called *associated vector bundles*.

6.1.3 Principal bundles

At this point, the question "why only take vector spaces as our fibers?" becomes even more pressing. One structure group can have many different representations, and so if we have a bunch of associated bundles, one is in no way more "fundamental" than the other. Is there a more generic object that describes the entire set of associated bundles at once?

Indeed there is, and this can be seen by realizing that there is a very natural object that G acts on – not a vector space, but the group G itself! The action we mean is simply left multiplication. (Or right multiplication, of course, but it is customary to choose the left one.) Thus, we can construct yet another associated bundle by starting from patches $U_i \times G$ and identifying

$$U_i \times G \ni (x, g_i) \sim (x, \phi_{ji}(x)g_i) \in U_j \times G. \tag{6.6}$$

Here, we slightly abused notation, as $\phi_{ji}(x)$ now really should be thought of as an element of the abstract group G, not as a matrix in some concrete representation.

Note that the bundle we have constructed here is no longer a *vector* bundle: its fiber is not a vector space, but some Lie group G. Such an object (constructed with the same glueing rules that vector bundles satisfy, $\phi_{ij}(x) = \phi_{ji}(x)^{-1}$, and so on) is called a *principal* bundle. The name *principal* is well chosen: it is the "generic" bundle that all associated vector bundles can be reduced to, and can be derived from.

6.1.4 Why principal bundles?

Recall that in lecture 4, we encountered a quantum field theory action of the following kind:

$$S = \int (\eta^{\mu\nu} D_\mu \phi^* D_\nu \phi - V(\phi^* \phi)) \tag{6.7}$$

Here, $\phi(x)$ was a complex scalar field, and

$$D_\mu \phi(x) = (\partial_\mu + A_\mu)\phi(x) \tag{6.8}$$

is the covariant derivative acting on this field[1]. By now, we know how to mathematically phrase what is going on here: A_μ is a connection on a vector bundle over space-time, and $\phi(x)$ is nothing but a section of that vector bunde. (So here, the vector bundle is simply a complex line bundle.)

This construction turns out to be exactly how the electromagnetic field (or the gauge fields for other fundamental forces like the strong and weak nuclear force, for that matter) interacts with the quantum fields describing matter particles. Of course, there is no charged scalar field in nature (as far as we know) that the electromagnetic field interacts with, but there are many other types of field for which it does: the fields describing electrons, quarks, muons, and so on. At this point, we are not able to fully describe these interactions yet, as all of these fields are *fermionic*, but apart from this subtlety that we will discuss in a lot of detail later on, the construction for all of those interactions is exacty as above.

However, in nature, there is only *one* electromagnetic field, but there are *many* particle fields that it interacts with! So it is not the case that each particle field is a section of some bundle which has its own, completely independent connection. In some sense, we want the connection (the gauge field) for each of the particle field bundles to be "the same" connection. But these particles may be described by objects with a totally different number of indices – in other words: they may be sections of bundles with totally different fibres. How can we still use one connection to describe all of these bundles?

The answer should be clear now: all of those bundles should be *associated* bundles, stemming from one, single principal bundle. The topology of this principal bundle, and the parallel transport in it, determines a "master connection", and all connections in the individual particle bundles should follow from this.

Thus, we need a principle bundle to make the idea that all charged particles are coupled to "the same gauge field" mathematically precise. The question now is, of course: can we also make the idea of such a "master connection" precise? We will soon see that the answer is "yes", but before doing so, let us mention a few more facts about and properties of principal bundles.

[1] In the definition of A we absorbed a factor of $-i$ compared to the notation in lecture 4.

6.1.5 Frame bundles, sections and triviality

Let us go back to the situation where the fiber is $\mathbb{R}^n$ and the structure group G is the full general linear group, $G = GL(n, \mathbb{R})$. (We focus on the real case now, but everything we say will hold for complex bundles as well.) In this case, there is an interesting construction of the associated principal bundle, called the *frame bundle*.

A *frame* for a vector bundle is a local choice of basis for the fiber. That is, to every $x \in M$, we associate an n-tuple of vectors

$$B(x) = \{e_1(x), e_2(x), \ldots, e_n(x)\} \tag{6.9}$$

such that this n-tuple forms a basis for F_x. Note that we have a collection of n vectors in an n-dimensional space, so the above expression is an $n \times n$ matrix. Moreover, the fact that it provides a basis means that the $e_i(x)$ are linearly independent, and so

$$\det B(x) \neq 0. \tag{6.10}$$

That is, $B(x)$ itself can be thought of as an element of $GL(n, \mathbb{R})$. Thus, the set of all frames over a point is isomorphic to $GL(n, \mathbb{R})$, and the reader probably won't be surprised that we can glue these $GL(n, \mathbb{R})$s together into a principal bundle: the *frame bundle*. We leave it as an exercise (see e.g. Nakahara) to show that the natural glueing maps for this frame bundle are exactly the glueing maps for the vector bundle that we started with, so that in this way we indeed construct the associated principal bundle.

The concept of a frame bundle leads us to a theorem that is as surprising as it is simple. One may wonder: is there an easy criterion to check whether a given vector bundle E is trivial – i.e. whether we can write it as $E \cong M \times F$? If a bundle is indeed trivial we should be able to find an explicit isomorphism from $M \times F$ to E – that is: we can view $M \times F$ as a trivial choice of *coordinates* on E. We can then look at images of coordinates of the form $(m, (1, 0, 0, \ldots, 0))$, for example. This should provide a section of E, and similarly the images of $(m, (0, 1, 0, \ldots, 0))$ provide a section, and so on. Continuing, we end up with n different sections, but these sections should not be arbitrary: if $M \times F$ is indeed isomorphic to E, we should be able to end up at any point in E by taking a linear combination of these sections. That is: the sections should all be *linearly independent*.

Thus, if our bundle has n linearly independent sections, the bundle is trivial, and of course the converse is also true: any trivial bundle has n linearly independent sections given by points of the form (m, e_i) where the e_i are some basis vectors of F. So a vector bundle is trivial *if and only if* it has n linearly independent global sections.

The above statement can be formulated much more easily in terms of frame bundles, as a collection of n linearly independent sections of the vector bundle is nothing but a global assignment of the $B(x)$ in (6.9) to any $x \in M$. Thus we arrive at the statement that *a vector bundle is trivial if and only if its associated frame bundle has a global section.*

There is one further generalization of this statement: it turns out to hold for *any* principal bundle, not just a frame bundle. That is, we have the following

Theorem: A principal bundle is trivial if and only if it has a global section.

The content of this theorem is rather surprising at first sight, but its proof is actually not that hard; you will prove the statement yourself in exercise 1 of this lecture. To get further acustomed to this surprising result, it may also be useful to realize that apparently, bundles do not always have global sections! (*Vector* bundles do; they always have the zero section.) For a very simple example, consider the boundary of the Möbius strip, which is a bundle whose fiber consists of two points, with base space S^1. Drawing a picture of this bundle, you can quickly see that it has no smooth global section. Similar topological obstructions to the existence of global section (though much harder to "see" in an image) occur in more complicated situations.

6.1.6 Connections on principal bundles

Recall from the previous lecture that there are two ways to describe a connection on a vector bundle E. The most "global" way is to use a linear map

$$\nabla : \Gamma(M, E) \to \Omega^1(M, E) \tag{6.11}$$

satisfying the Leibniz rule. The connection ∇ describes the directional derivatives of a section ("function") $s \in \Gamma(M, E)$: it outputs a one-form on M with values in E, that is: an object which we can locally (at $x \in M$) insert a tangent vector into, and which then gives the derivative in that tangent direction as an element in the fiber.

The second description of a connection is much more in the "physics sense": we have seen that locally (on a patch $U_i \subset M$), one can always write the connection as $\nabla = d + A_i$, where d is the exterior derivative and A is a Lie-algebra valued one form. (In the case of vector bundles, with structure group $GL(n, \mathbb{R})$, this Lie algebra is simply the space of all $n \times n$ matrices.) However, this construction only works *locally*: over a different patch U_j, we can again write $\nabla = d + A_j$, but on an overlap, A_i and A_j need not be the same: they are generically connected by a gauge transformation,

$$A_i = \phi_{ij} A_j \phi_{ji}^{-1} - \phi_{ij}^{-1} d\phi_{ij}. \tag{6.12}$$

In fact, we have already encountered this issue very early on, when we needed to describe the Dirac monopole using two different gauge fields on two different patches.

Fortunately, in the associated frame bundle, things are a lot nicer. Note that using the projection map $\pi : F(E) \to M$ (where $F(E)$ is the frame bundle associated to E), one can pull back the Lie algebra valued one-forms A_i to a Lie algebra valued one-form $\pi^* A_i$ on (part of) $F(E)$. Similarly, one can pull back A_j. Now here comes the nice thing: a (nontrivial) computation now shows that

$$\pi^* A_i = \pi^* A_j, \tag{6.13}$$

that is, the two pullbacks *are* the same, without the need to do any gauge transformations! Thus we have finally arrived at the true, global description of our gauge field A (without the need to introduce any derivations like ∇): A should be viewed as a Lie algebra valued 1-form, defined on the principal bundle associated to our vector bundle(s).

For completeness, let us note that not any Lie algebra valued 1-form on a principal bundle will do. To define a good connection, the 1-form needs to have nice transformation properties when we move along the fiber, for example. More details about this can be found in the mathematics notes for the previous lecture, as well as in section 10.1 of Nakahara.

6.2 Instantons

6.2.1 Instantons in Yang-Mills theory

We now want to turn to some examples where bundles and (particularly) connections play an important role. The first of these examples is the concept of an *instanton*. Recall that in quantum field theory (and in classical field theory, for that matter), we are interested in finding field configurations which minimize the action $S[\phi] = \int d^4x L(\phi(x), \partial_\mu \phi(x))$. We encountered one way to find such configurations: by solving the Euler-Lagrange equations

$$\frac{\partial L}{\partial \phi} = \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}. \tag{6.14}$$

However, solving these equations is generally not straightforward: they are second (or higher) order differential equations for $\phi(x)$. Fortunately, sometimes there are easier ways to find minima of the action, and this is in particular true for Yang-Mills theory. For now, let us assume that we are in the Euclidean setting, so that we treat space-time really as a four-dimensional space, with all coordinates on an equal footing and no minus signs coming from an indefinite metric. In such a situation, solutions to the (Euclidean) equations of motion are called *instantons*. We want to find such instantons for our Yang-Mills theory action:

$$S = \int F \wedge \star F \tag{6.15}$$

Note that to simplify notation, we implicitly assumed the trace operation in our notation here. How can we find minima of this action? To do so, let us recall that the inner product $(\alpha, \beta) = \int \alpha \wedge \star\beta$ is a nondegenerate and (in the Euclidean case) positive definite inner product on the space of (here: Lie-algebra valued) two-forms. Now consider the following integrals:

$$T_\pm = \int (F \pm \star F) \wedge \star (F \pm \star F), \tag{6.16}$$

where we choose the same sign in both factors. Since our inner product is positive definite, we have that $T_\pm \geq 0$. Moreover, by using symmetry of our inner product and the fact that on Euclidean two-forms the Hodge star squares to 1, we find that

$$2S \pm 2 \int F \wedge F \geq 0 \tag{6.17}$$

or

$$S \geq \left| \int F \wedge F \right|. \tag{6.18}$$

From the derivation, we see that equality is obtained whenever $F = \pm \star F$. Field configurations satisfying this condition are called *(anti-) selfdual* field configurations. To be explicit, let us focus on the selfdual case, where $F = \star F$.

Why are these configurations so interesting? The reason is that we will see (in a moment in a particular example, and in much more detail in lectures that follow) that $\int F \wedge F$ is a very special object. It is what is called a *characteristic class*, and it turns out to be a discrete topological invariant of the bundle that A is a connection for. "Discrete" here means that $\int F \wedge F$ is *quantized*: up to a normalization factor, it can only take integer values. In particular, that means that small changes in F do *not* change the value of $\int F \wedge F$. And that in turn means that the solutions for which (6.18) is realized, are local minima of the action! That is, the selfdual field configurations are instantons: solutions to the Euclidean equations of motion!

(A different way to see this: recall that F by construction satisfies the Bianchi identity $dF = 0$. However, the equations of motion can be written as $d \star F = 0$. Clearly, if $F = \star F$, one implies the other.)

6.2.2 The $SU(2)$ case

For concreteness, let us study the case where our gauge group is $SU(2)$. The group $SU(2)$ as a manifold in fact equals S^3. To see this, introduce coordinates (y^1, y^2, y^3, y^4) for S^3, with $\sum (y^i)^2 = 1$. Then it is not too hard to see (see exercise 2) that an arbitrary element of $SU(2)$ can be written as

$$U = i t^i \sigma_i + t^4 I_2 \tag{6.19}$$

with I_2 the 2×2 identity matrix, and σ_i the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.20}$$

Now, we would like to construct an explicit instanton on an $\mathbb{R}^4$ spacetime. In fact, we would like our field configuration to be trivial far away from the origin: we divide spacetime into two regions:

$$U_N = \{x \in \mathbb{R}^4 || x| < R + \epsilon\}, \qquad U_S = \{x \in \mathbb{R}^4 || x| > R - \epsilon\} \tag{6.21}$$

for some large R. (In fact, we have to take $R \to \infty$ to have the arguments below work exactly.) The patch U_N is topologically trivial. U_S is not, but since we want F to be vanishing on U_S anyway, we may as well add the point at infinity (where F remains zero) and consider our field configuration as a configuration on S^4 instead of R^4, so that U_N and U_S are its northern and souther hemispheres.

As we mentioned, we want our fields to be trivial far away from the origin: over U_S, we pick a local connection $A_S = 0$. On the overlap of U_N and U_S (homotopy equivalent to the "equator" S^3), this means that to view A as a good connection on a bundle, we have to take it to be "pure gauge" on the northern hemisphere:

$$A_N = U(x)^{-1} dU(x). \tag{6.22}$$

Which $U(x)$ can we take? As the overlap of U_N and U_S is essentially (up to the "thickness" 2ϵ) an S^3 itself, and $SU(2)$ is also an S^3, a natural map could be the identity mapping between these to S^3s. That is, we want to take

$$U(x) = \frac{1}{r}\left(ix^i\sigma_i + x^4 I_2\right) \tag{6.23}$$

where $r^2 = \sum (x^i)^2$. This gives a nice connection A_N on the equator, but can it also be extended over the entire northern hemisphere? Here, the above definition will not do: it clearly is invariant under an overall scaling of all the x^i, and so the answer is not well defined (it depends on the direction) if we send $x \to 0$. It turns out that a good extension is to define

$$A_N = f(r) U(x)^{-1} dU(x) \tag{6.24}$$

with $f(r)$ a function that vanishes at $r = 0$ and that becomes 1 for large r. In fact, so far we have not imposed our requirement that $F = \star F$ yet. Imposing it, one can compute what $f(r)$ must be; the answer turns out to be that

$$f(r) = \frac{r^2}{r^2 + c^2} \tag{6.25}$$

with c a free parameter that can be thought of as the "size" of the (nontrivial part of the) instanton.

Now what is the value of $\int F \wedge F$ for this instanton? In exercise 2, the answer is computed in two steps. First of all, you will show that

$$\int_{S^4} F \wedge F = -\frac{1}{3} \int_{S^3} A \wedge A \wedge A, \tag{6.26}$$

where the second integral is over the equator. Then, you will show that the integrand is proportional to the volume 3-form ω on the sphere:

$$\int_{S^3} A \wedge A \wedge A = 12 \int_{S^3} \omega \tag{6.27}$$

Using the fact that the volume of a three-sphere is $2\pi^2$, we arrive at the result that

$$\int_{S^4} F \wedge F = 8\pi^2. \tag{6.28}$$

Using more elaborate computation, one can find other instanton solutions generalizing (6.23), for example by starting from

$$U(x) = \frac{1}{r^n}\left(ix^i\sigma_i + x^4 I_2\right)^n, \tag{6.29}$$

a map which "wraps" the equator S^3 of S^4 around $SU(2) = S^3$ not once, but n times. It turns out that for these configurations,

$$\int_{S^4} F \wedge F = 8\pi^2 n. \tag{6.30}$$

Thus, we see that $\int F \wedge F$ indeed measures something topological: it measures (when properly normalized) the "wrapping number" of the instanton. In particular, one can show very generally that it satisfies the above quantization rule. We will not prove this result here – it is simply a consequence of the much more general statements about characteristic classes that we will make later.

6.3 The Berry phase

So far, we have been discussing physics situations where space-time itself is topologically nontrivial: the Dirac monopole, the Aharonov-Bohm effect and the instanton configurations discussed in the previous section. However, in most practical sitations, space-time is simply Minkowski-space, or a curved version thereof, without nontrivial topological features. Therefore, the reader may have started wondering how useful our topological methods really are in practice. To ease such a worried reader's mind, in this section we will study a different application of topology in physics, where it is not *spacetime* which is topologically trivial, but a *configuration* space. We will see that in this situation, all of the same concepts play a role.

In fact, we will not specify our configuration space in much detail. (Recall, however, from the exercises for lecture 1, that such configuration spaces, even for simple classical mechanical systems, can have pretty much any topology we want.) We will simply assume that our configuration space is parameterized by some parameters that we will collectively denote by X.

Now, we are interested in the following problem. We want to find energy eigenstates of an X-dependent hamiltonian, but in a setting where X changes slowly as a function of time. That is, we want to study the following time-dependent Schrödinger equation:

$$H(X(t))|\psi(t)\rangle = i\frac{d}{dt}|\psi(t)\rangle \tag{6.31}$$

where as usual, we have set $\hbar = 1$. Moreover, suppose that we have already solved the eigenvalue problem for any fixed value of X. That is, we know the eigenvalues $E_n(X)$ and the eigenstates $|n, X\rangle$ solving the equation

$$H(X)|n, X\rangle = E_n(X)|n, X\rangle \tag{6.32}$$

Finally, we assume that our changes in X are small enough that different eigenvalues $E_n(X)$ never cross. Can we use this information to construct $|\psi(t)\rangle$?

Formally, one could solve (6.31) by integrating and exponentiating the operator $H(R(t))$. Attempting to do this at the level of a single eigenvalue might lead one to a guess of the form

$$|\psi(t)\rangle \approx \exp\left(-i\int_0^t E_n(X(s))ds\right)|n, X(t)\rangle. \tag{6.33}$$

However, it is clear that this does not quite work: taking the time derivative of the right hand side does not just get the prefactor $E_n(X(t))$ from differentiating the first factor, it also leads to a derivative of the state itself:

$$i\frac{d}{dt}|\psi(t)\rangle = E_n(X(t))|\psi(t)\rangle + i\exp\left(-i\int_0^t E_n(X(s))ds\right)\frac{d}{dt}|n, X(t)\rangle \tag{6.34}$$

whereas acting with $H(X(t))$ on $|\psi(t)\rangle$ only gives the first term.

It was Michael Berry who observed that the above problem can be solved by adding an additional phase to the construction of $|\psi(t)\rangle$:

$$|\psi(t)\rangle = \exp\left(i\eta(t) - i\int_0^t E_n(X(s))ds\right)|n, X(t)\rangle. \tag{6.35}$$

Taking this phase into account leads to an extra term when taking the t-derivative, and so the Schrödinger equation now has two leftover terms when we plug in $|\psi(t)\rangle$:

$$0 = -\frac{d\eta(t)}{dt}|\psi(t)\rangle + i\exp\left(i\eta(t) - i\int_0^t E_n(X(s))ds\right)\frac{d}{dt}|n, X(t)\rangle. \tag{6.36}$$

We now take the inner product of this equation with $\langle n, X(t)|$ and obtain

$$\frac{d\eta(t)}{dt} = i\,\langle n, X(t)|\frac{d}{dt}|n, X(t)\rangle. \tag{6.37}$$

Before continuing with this expression, note that the right hand side is real: its complex conugate is

$$-i\,\left(\frac{d}{dt}\langle n, X(t)|\right)|n, X(t)\rangle \tag{6.38}$$

and subtracting this from the previous expression we het the time derivative of

$$\langle n, X(t)|n, X(t)\rangle = 1, \tag{6.39}$$

which clearly vanishes. Thus, $d\eta/dt$ equals its complex conjugate: it is real, and so we can indeed speak of an added *phase* in our quantum state.

Integrating our result now gives an expression for $\eta(t)$ itself:

$$\eta(t) = i \int_0^t \langle n, X(s)| \frac{d}{ds} |n, X(s)\rangle \tag{6.40}$$

which after a change of variables we may also write as

$$\eta(t) = i \int_{X(0)}^{X(t)} \langle n, X| \frac{d}{dX} |n, X\rangle dX. \tag{6.41}$$

Note that the integrand is not a total derivative itself, so that even when $X(0) = X(t)$, the phase may not vanish. That is: the phase does not just depend on the point in configuration space, it also depends on how that particular configuration was obtained. Moreover, it explicitly depends on the energy level n, so if we start from a certain configuration and go through a topologically nontrivial loop, different energy eigenstates will obtain different phases.

6.3.1 Rephrasing in bundle language

The above can be rephrased in a language which makes use of the bundles and connections that we have seen so far. First of all, note that any state $|\psi\rangle$ is physically equivalent to the state $e^{i\phi}|\psi\rangle$. In other words, we can view any space of states as a $U(1)$ bundle over a smaller base space M:

$$M = \{\text{states}\}/\sim \tag{6.42}$$

where $\sim$ is the equivalence mentioned above. We should think of our parameter(s) X introduced above as coordinates on this M, and as the states $|n, X\rangle$ as sections of the $U(1)$-bundle. The phase is then the integral of a connection obtained from this section:

$$A = \langle n, X| \cdot d|n, X\rangle \tag{6.43}$$

To see that this makes sense as a connection on a principal $U(1)$ bundle, note that the ket vector $\langle n, X|$ indeed has the opposite phase of the corresponding bra vector, so that this is precisely the way in which one defines a connection. With some more work, one can show that indeed, A satisfies all further conditions for a connection one-form on a principal bundle.

What is the physical relevance of this Berry connection? We will see an example in exercise 3, where you will show that in certain physical problems involving "slow" and "fast" degrees of freedom, the latter can be integrated out of the problem – at the expense that the hamiltonian for the slow degrees of freedom now must be phrased in terms of *covariant* derivatives, where the relevant connection is precisely the Berry connection.